The Variational Iteration Method for a Class of Eighth-Order Boundary-Value Differential Equations

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The variational iteration method, a well-known method for solving functional equations, is employed to solve a class of eighth-order boundary-value problems, which govern scientific and engineering experimentations. Some special cases of the mentioned equations are solved as examples to illustrate the ability and reliability of the method. The results reveal that the method is very effective and convenient.

Key words: Variational Iteration Method; Eighth-Order Boundary-Value Problems.

1. Introduction

The solution of nonlinear problems by analytical techniques is often rather difficult. Higher-order initial-boundary-value problems arise in many engineering applications [1 – 5]. Various numerical and analytical methods were proposed to solve such problems (see [6 – 10] and references cited therein). Eighth-order differential equations govern physics of some hydrodynamic stability problems. When an infinite horizontal layer of fluid is heated from below and is subjected to rotation, instability is observed. When this instability starts as an overstability, it is modelled by an eighth-order ordinary differential equation [11]. Siddiqi and Twizell [12] used polynomial splines of degree six to solve some special cases of linear eighth-order boundary-value problems. However, the obtained results were divergent at points adjacent to the boundary. Twizell and his coworkers [13, 14] have also solved some other higher-order problems and encountered the same problems. Liu and Wu [15], utilizing the generalized differential quadrature rule, have solved similar problems.

In this paper, a class of eighth-order boundary-value problems is considered. Four examples, solved in [12, 15], are dealt with again to obtain accurate results in the entire domain via the variational iteration method. We consider the following class of eighth-order boundary-value problems:

$$u^{(8)}(x)+\phi(x)u(x)=\psi(x), \quad a\le x\le b, \tag{1}$$

with the boundary conditions

$$\begin{aligned}&u(a)=A_0, && u^{(2)}(a)=A_2, && u^{(4)}(a)=A_4,\\ &u^{(6)}(a)=A_6, && u(b)=B_0, && u^{(2)}(b)=B_2,\\ &u^{(4)}(b)=B_4, && u^{(6)}(b)=B_6,\end{aligned}$$

where $u=u(x)$ and $\phi(x)$ and $\psi(x)$ are continuous functions defined on $[a,b]$ and the constants A_i and B_i are finite real numbers. Furthermore, we will apply the variational iteration method on the following example [16]:

$$u^{(8)}(x)=\mathrm{e}^{-x}u^2(x), \quad 0\le x\le 1, \tag{2}$$

with the boundary conditions

$$\begin{aligned}&u(0)=u^{(2)}(0)=u^{(4)}(0)=u^{(6)}(0)=1,\\ &u(1)=u^{(2)}(1)=u^{(4)}(1)=u^{(6)}(1)=\mathrm{e}.\end{aligned}$$

The article is organized as follows: Section 2 briefly reviews the mathematical basis of the variational iteration method used for this study. Five illustrative examples are documented in Section 3. These examples intuitively describe the ability and reliability of the method. A conclusion and future directions for research are all summarized in the last section.

2. The Variational Iteration Method

The basic concept of the variational iteration method is illustrated in [17 – 22], and applications of the

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method can be found in [6–8, 23–31]. We consider the general nonlinear system

$$L[u(x)]+R[u(x)]+N[u(x)]=\psi(x),$$

where $L=\frac{\mathrm{d}^8}{\mathrm{d}x^8}$, R is a linear operator, N the nonlinear term and $\psi(x)$ a given continuous function. According to the variational iteration method, we can construct a correction functional in the form

$$\begin{aligned}u_{n+1}(x)=u_n(x)+\int_0^x \lambda(s)[Lu_n(s)+R\tilde{u}_n(s)\\+N\tilde{u}_n(s)-\psi(s)]\mathrm{d}s,\end{aligned}$$

where $u_0(x)$ is an initial approximation with possible unknowns, λ is a Lagrange multiplier which can be identified optimally via the variational theory, the subscript n denotes the nth approximation, and $\tilde{u}_n$ is considered as a restricted variation, i. e., $\delta\tilde{u}_n=0$.

For eighth-order boundary-value problems mentioned above, according to the variational iteration method, the nonlinear terms have to be considered as restricted variation. So we derive a correction functional as follows:

$$\begin{aligned}u_{n+1}(x)=u_n(x)+\int_0^x \lambda(s)[u^{(8)}(s)\\+\phi(s)\tilde{u}(s)-\psi(s)]\mathrm{d}s.\end{aligned}$$

The stationary condition of the above correction functional can be expressed as

$$\begin{aligned}&\lambda^{(8)}(s)=0,\\&1-\lambda^{(7)}(s)|_{s=x}=0,\\&\lambda^{(i)}(s)|_{s=x}=0,\quad i=1,2,\ldots,6,\\&\lambda(s)|_{s=x}=0.\end{aligned}$$

The Lagrange multiplier, therefore, can be identified as

$$\lambda=\frac{1}{7!}(s-x)^7.$$

As a result we obtain the iteration formula

$$\begin{aligned}u_{n+1}(x)=u_n(x)+\frac{1}{7!}\int_0^x (s-x)^7[u^{(8)}(s)\\+\phi(s)u(s)-\psi(s)]\mathrm{d}s.\end{aligned}\tag{3}$$

In the above-mentioned iteration formula, we begin with an arbitrary initial approximation:

$$u_0(x)=a+bx+cx^2+dx^3+kx^4+fx^5+gx^6+hx^7,$$

where a, b, c, d, k, f, g and h are constants to be determined.

3. Applications

In this section, we present five examples to show the efficiency and high accuracy of the present method.

3.1. Example 1

Consider (1) with $[a,b]=[0,1]$, $\phi(x)=x$, $\psi(x)=-(48+15x+x^3)\mathrm{e}^x$, and the boundary conditions [12, 15]

$$\begin{aligned}&A_0=0,\quad A_2=0,\quad A_4=-8,\quad A_6=-24,\\&B_0=0,\quad B_2=-4\mathrm{e},\quad B_4=-16\mathrm{e},\quad B_6=-36\mathrm{e}.\end{aligned}\tag{4}$$

According to (3) we have the following iteration formula:

$$\begin{aligned}u_{n+1}(x)=u_n(x)+\frac{1}{7!}\int_0^x (s-x)^7[u_n^{(8)}(s)\\+su_n(s)+(48+15s+s^3)\mathrm{e}^s]\mathrm{d}s.\end{aligned}\tag{5}$$

Now, we begin with the arbitrary initial approximation

$$u_0(x)=a+bx+cx^2+dx^3+kx^4+fx^5+gx^6+hx^7,$$

where a, b, c, d, k, f, g and h are constants to be determined.

By the variational iteration formula (5), we have

$$\begin{aligned}u_1(x)&=u_0(x)+\frac{1}{7!}\int_0^x (s-x)^7[u_0^{(8)}(s)+su_0(s)\\&\qquad+(48+15s+s^3)\mathrm{e}^s]\mathrm{d}s\\&=a-792+(b-561)x+(c-189)x^2+\left(d-\frac{79}{2}\right)x^3\\&+\left(k-\frac{11}{2}\right)x^4+\left(f-\frac{19}{40}\right)x^5+\left(g-\frac{1}{120}\right)x^6\\&+\left(h+\frac{3}{560}\right)x^7-\frac{ax^9}{362880}-\frac{bx^{10}}{1814400}-\frac{cx^{11}}{6652800}\\&-\frac{dx^{12}}{19958400}-\frac{kx^{13}}{51891840}-\frac{fx^{14}}{121080960}-\frac{gx^{15}}{259459200}\\&-\frac{hx^{16}}{518918400}+792\mathrm{e}^x-231x\mathrm{e}^x+24x^2\mathrm{e}^x-x^3\mathrm{e}^x.\end{aligned}$$

Incorporating the boundary conditions (4) into $u_1(x)$ yields a linear system with 8 equations and 8 variables. Solving this linear system simultaneously, we have $a=0$, $b=1$, $c=0$, $d=\frac{-1}{2}$, $k=\frac{-1}{3}$, $f=\frac{-1}{8}$, $g=\frac{-1}{30}$ and $h=\frac{-1}{144}$. Thus we obtain the first-order approximate solution

$$u_1(x)=x-\frac{x^3}{2}-\frac{x^4}{3}-\frac{x^5}{8}-\frac{x^6}{30}-\frac{x^7}{144}.$$

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Table 1. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one.

x	Exact solution	First-order approximate solution	Absolute error
0	0	0	0
0.1	0.9946538262	0.9946538265	$\leq 10^{-9}$
0.2	0.1954244413	0.1954244445	$\leq 10^{-8}$
0.3	0.2834703497	0.2834704312	$\leq 10^{-6}$
0.4	0.3580379275	0.3580387556	$\leq 10^{-5}$
0.5	0.4121803178	0.4121853299	$\leq 10^{-5}$
0.6	0.4373085120	0.4373304000	$\leq 10^{-4}$
0.7	0.4228880685	0.4229643785	$\leq 10^{-3}$
0.8	0.3560865485	0.3563121778	$\leq 10^{-3}$
0.9	0.2213642800	0.2219525438	$\leq 10^{-3}$
1	0	0.0013888888	$\leq 10^{-2}$

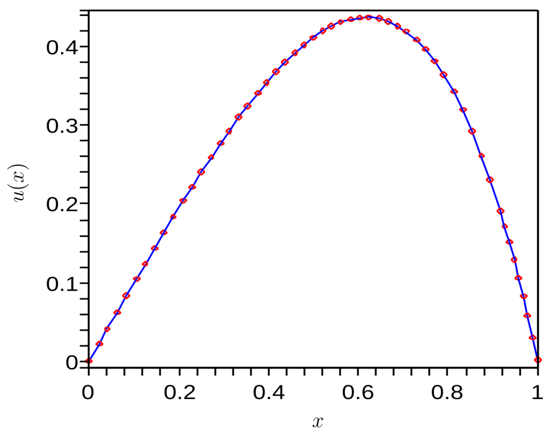

Fig. 1. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one. Solid line, exact solution; symbols, approximate solution.

As the same, we find $u_2(x)$ as follows:

$$\begin{aligned}u_2(x) &= u_1(x) + \frac{1}{7!}\int_0^x (s-x)^7 [u_1^{(8)}(s) + su_1(s) \\ &\qquad + (48+15s+s^3)\mathrm{e}^s]\mathrm{d}s \\ &= x - \frac{x^3}{2} - \frac{x^4}{3} - \frac{x^5}{8} - \frac{x^6}{30} - \frac{x^7}{144} - \frac{x^8}{840} - \frac{x^9}{5760} \\ &\quad - \frac{x^{10}}{45360} - \frac{x^{11}}{403200} - \frac{x^{12}}{3991680} - \frac{x^{13}}{43545600} \\ &\quad - \frac{x^{14}}{518918400} - \frac{x^{15}}{6706022400} + O(x^{16}).\end{aligned}$$

This gives the solution in a closed form by $x(1-x)\mathrm{e}^x$. The first-order approximate solution is of remarkable accuracy. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one is shown in Table 1 and Figure 1.

3.2. Example 2

Consider (1) with $[a,b] = [-1,1]$, $\phi(x) = -x$ and $\psi(x) = -(55+17x+x^2-x^3)\mathrm{e}^x$, with the boundary conditions [12, 15]

$$\begin{aligned}&A_0 = 0,\ \ A_2 = 2/\mathrm{e},\ \ A_4 = -4/\mathrm{e},\ \ A_6 = -18/\mathrm{e}, \\ &B_0 = 0,\ \ B_2 = -6\mathrm{e},\ \ B_4 = -20\mathrm{e},\ \ B_6 = -42\mathrm{e}.\end{aligned} \tag{6}$$

According to (3) we have the following iteration formula:

$$\begin{aligned}u_{n+1}(x) = u_n(x) + \frac{1}{7!}\int_0^x (s-x)^7 [u_n^{(8)}(s) - su_n(s) \\ + (55+17s+s^2-s^3)\mathrm{e}^s]\mathrm{d}s.\end{aligned} \tag{7}$$

Now, we begin with the arbitrary initial approximation

$$u_0(x) = a + bx + cx^2 + dx^3 + kx^4 + fx^5 + gx^6 + hx^7,$$

where a, b, c, d, k, f, g and h are constants to be determined. By the variational iteration formula (7), we have

$$\begin{aligned}u_1(x) &= u_0(x) + \frac{1}{7!}\int_0^x (s-x)^7 [u_0^{(8)}(s) - su_0(s) \\ &\qquad + (55+17s+s^2-s^3)\mathrm{e}^s]\mathrm{d}s \\ &= 711 + a + (496+b)x + \left(c + \frac{331}{2}\right)x^2 + (d+35)x^3 \\ &\quad + \left(k + \frac{127}{24}\right)x^4 + \left(f + \frac{19}{30}\right)x^5 + \left(g + \frac{17}{240}\right)x^6 \\ &\quad + \left(h + \frac{23}{2520}\right)x^7 + \frac{ax^9}{362880} + \frac{bx^{10}}{1814400} + \frac{cx^{11}}{6652800} \\ &\quad + \frac{dx^{12}}{19958400} + \frac{kx^{13}}{51891840} + \frac{fx^{14}}{121080960} + \frac{gx^{15}}{259459200} \\ &\quad + \frac{hx^{16}}{518918400} - 25\mathrm{e}^x x^2 + 215\mathrm{e}^x x - 711\mathrm{e}^x + \mathrm{e}^x x^3.\end{aligned}$$

Incorporating the boundary conditions (6) into $u_1(x)$ yields a linear system with 8 equations and 8 variables. Solving this linear system simultaneously, we have $a = 1$, $b = 1$, $c = \frac{-1}{2}$, $d = \frac{-5}{6}$, $k = \frac{-11}{24}$, $f = \frac{-19}{120}$, $g = \frac{-29}{720}$ and $h = \frac{-41}{5040}$. Thus we obtain the first-order approximate solution

$$\begin{aligned}u_1(x) = 1 + x - \frac{x^2}{2} - \frac{5x^3}{6} - \frac{11}{24}x^4 - \frac{19}{120}x^5 \\ - \frac{29}{720}x^6 - \frac{41}{5040}x^7.\end{aligned}$$

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Table 2. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one.

x	Exact solution	First-order approximate solution	Absolute error
-1	0	0.1190476190	$\leq 10^{-2}$
-0.8	0.1617584271	0.1619634388	$\leq 10^{-3}$
-0.6	0.3512394471	0.3512605257	$\leq 10^{-4}$
-0.4	0.5630688386	0.5630696838	$\leq 10^{-5}$
-0.2	0.7859815230	0.7859815264	$\leq 10^{-8}$
0	1	1	0
0.2	1.172546648	1.172546651	$\leq 10^{-7}$
0.4	1.253132746	1.253133695	$\leq 10^{-5}$
0.6	1.166156032	1.166181074	$\leq 10^{-4}$
0.8	0.8011947341	0.8014527387	$\leq 10^{-3}$
1	0	0.1587301587	$\leq 10^{-2}$

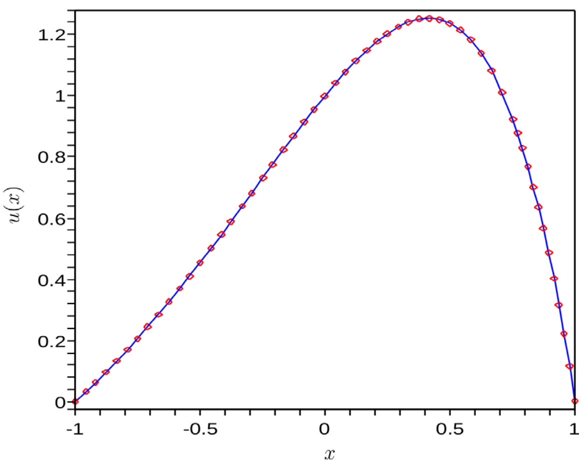

Fig. 2. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one. Solid line, exact solution; symbols, approximate solution.

As the same, we find $u_2(x)$ as follows:

$$\begin{aligned} u_2(x) &= u_1(x) + \frac{1}{7!}\int_0^x (s-x)^7 [u_1^{(8)}(s) - s u_1(s) \\ &\quad + (55+17s+s^2-s^3)\mathrm{e}^s]\mathrm{d}s \\ &= 1 + x - \frac{x^2}{2} - \frac{5x^3}{6} - \frac{11x^4}{24} - \frac{19x^5}{120} - \frac{29x^6}{720} - \frac{41x^7}{5040} \\ &\quad - \frac{11x^8}{8064} - \frac{71x^9}{362880} - \frac{89x^{10}}{3628800} - \frac{109x^{11}}{39916800} \\ &\quad - \frac{131x^{12}}{479001600} - \frac{31x^{13}}{1245404160} - \frac{181x^{14}}{87178291200} \\ &\quad - \frac{19x^{15}}{118879488000} + O(x^{16}). \end{aligned}$$

This gives the solution in a closed form by $(1-x^2)\mathrm{e}^x$. The first-order approximate solution is of remarkable accuracy. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one is shown in Table 2 and Figure 2.

3.3. Example 3

Consider (1) with $[a,b] = [-1,1]$, $\phi(x) = -1$ and $\psi(x) = -8[2x \cos(x) + 7\sin(x)]$, with the boundary conditions [12, 15]

$$\begin{aligned} &A_0 = 0, \quad A_2 = -4\cos(1) - 2\sin(1), \\ &A_4 = 8\cos(1) + 12\sin(1), \\ &A_6 = -12\cos(1) - 30\sin(1), \\ &B_0 = 0, \quad B_2 = 4\cos(1) + 2\sin(1), \\ &B_4 = -8\cos(1) - 12\sin(1), \\ &B_6 = 12\cos(1) + 30\sin(1). \end{aligned} \tag{8}$$

According to (3) we have the iteration formula

$$\begin{aligned} u_{n+1}(x) = u_n(x) + \frac{1}{7!}\int_0^x (s-x)^7 \{u_n^{(8)}(s) - u_n(s) \\ + 8[2s\cos(s) + 7\sin(s)]\}\mathrm{d}s. \end{aligned} \tag{9}$$

Now, we begin with the arbitrary initial approximation

$$u_0(x) = a + bx + cx^2 + dx^3 + kx^4 + fx^5 + gx^6 + hx^7,$$

where a, b, c, d, k, f, g and h are constants to be determined. By the variational iteration formula (9), we have

$$\begin{aligned} u_1(x) &= u_0(x) + \frac{1}{7!}\int_0^x (s-x)^7 \{u_0^{(8)}(s) - u_0(s) \\ &\quad + 8[2s\cos(s) + 7\sin(s)]\}\mathrm{d}s \\ &= a + (b-56)x + cx^2 + (d+4)x^3 + kx^4 + \left(f + \frac{1}{15}\right)x^5 \\ &\quad + gx^6 + \left(h - \frac{1}{126}\right)x^7 + \frac{ax^8}{40320} + \frac{bx^9}{362880} \\ &\quad + \frac{cx^{10}}{1814400} + \frac{dx^{11}}{6652800} + \frac{kx^{12}}{19958400} + \frac{fx^{13}}{51891840} \\ &\quad + \frac{gx^{14}}{121080960} + \frac{hx^{15}}{259459200} - 16\cos(x)x + 72\sin(x). \end{aligned}$$

Incorporating the boundary conditions (8) into $u_1(x)$ yields a linear system with 8 equations and 8 variables. Solving this linear system simultaneously, we have $a = 0$, $b = -1$, $c = 0$, $d = \frac{7}{6}$, $k = 0$, $f = \frac{7}{40}$, $g = 0$ and $h = \frac{43}{5040}$. Thus we obtain the first-order approximate solution

$$u_1(x) = -x + \frac{7x^3}{6} - \frac{7}{40}x^5 + \frac{43}{5040}x^7.$$

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Table 3. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one.

x	Exact solution	First-order approximate solution	Absolute error
-1	0	-0.1984126984	$\le 10^{-3}$
-0.8	0.2582481927	0.2582214298	$\le 10^{-4}$
-0.6	0.3613711830	0.3613691657	$\le 10^{-4}$
-0.4	0.3271114075	0.3271113549	$\le 10^{-6}$
-0.2	0.1907225576	0.1907225575	$\le 10^{-9}$
0	0	0	0
0.2	-0.1907225576	-0.1907225575	$\le 10^{-9}$
0.4	-0.3271114075	-0.3271113549	$\le 10^{-6}$
0.6	-0.3613711830	-0.3613691657	$\le 10^{-4}$
0.8	-0.2582481927	-0.2582214298	$\le 10^{-4}$
1	0	0.1984126984	$\le 10^{-3}$

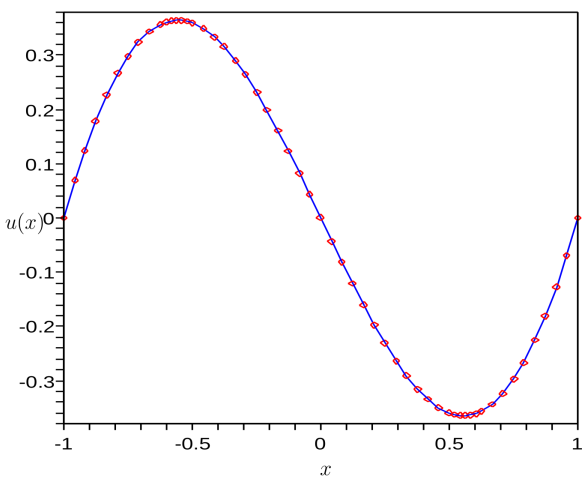

Fig. 3. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one. Solid line, exact solution; symbols, approximate solution.

Similarly we can find $u_2(x)$ as follows:

$$\begin{aligned} u_2(x) &= u_1(x) + \frac{1}{7!}\int_0^x (s-x)^7\{u_1^{(8)}(s) - u_1(s) \\ &\qquad +8[2s\cos(s)+7\sin(s)]\}\mathrm{d}s \\ &= -x + \frac{7x^3}{6} - \frac{7x^5}{40} + \frac{43x^7}{5040} - \frac{73x^9}{362880} + \frac{37x^{11}}{13305600} \\ &\quad - \frac{157x^{13}}{6227020800} + O(x^{15}). \end{aligned}$$

This gives the solution in a closed form by $(x^2-1)\sin(x)$. The first-order approximate solution is of remarkable accuracy. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one is shown in Table 3 and Figure 3.

3.4. Example 4

Consider (1) with $[a,b] = [-1,1]$, $\phi(x) = -1$ and $\psi(x) = 8[2x\sin(x) - 7\cos(x)]$, with the boundary conditions [12, 15]

$$\begin{aligned} &A_0 = 0, \quad A_2 = -4\sin(1) + 2\cos(1), \\ &A_4 = 8\sin(1) - 12\cos(1), \\ &A_6 = -12\sin(1) + 30\cos(1), \\ &B_0 = 0, \quad B_2 = -4\sin(1) + 2\cos(1), \\ &B_4 = 8\sin(1) - 12\cos(1), \\ &B_6 = -12\sin(1) + 30\cos(1). \end{aligned} \tag{10}$$

According to (3) we have the iteration formula

$$\begin{aligned} u_{n+1}(x) = u_n(x) + \frac{1}{7!}\int_0^x (s-x)^7\{u_n^{(8)}(s) - u_n(s) \\ - 8[2s\sin(s) - 7\cos(s)]\}\mathrm{d}s. \end{aligned} \tag{11}$$

Now, we begin with the arbitrary initial approximation

$$u_0(x) = a + bx + cx^2 + dx^3 + kx^4 + fx^5 + gx^6 + hx^7,$$

where a, b, c, d, k, f, g and h are constants to be determined. By the variational iteration formula (11), we have

$$\begin{aligned} u_1(x) &= u_0(x) + \frac{1}{7!}\int_0^x (s-x)^7\{u_0^{(8)}(s) - u_0(s) \\ &\qquad -8[2s\sin(s) - 7\cos(s)]\}\mathrm{d}s \\ &= -72 + a + bx + (c+20)x^2 + dx^3 + \left(k - \frac{1}{3}\right)x^4 \\ &\quad + fx^5 + \left(g - \frac{1}{30}\right)x^6 + hx^7 + \frac{ax^8}{40320} + \frac{bx^9}{362880} \\ &\quad + \frac{cx^{10}}{1814400} + \frac{dx^{11}}{6652800} + \frac{kx^{12}}{19958400} + \frac{fx^{13}}{51891840} \\ &\quad + \frac{gx^{14}}{121080960} + \frac{hx^{15}}{259459200} + 16\sin(x)x + 72\cos(x). \end{aligned}$$

Incorporating the boundary conditions (10) into $u_1(x)$ yields a linear system with 8 equations and 8 variables. Solving this linear system simultaneously, we have $a = -1, b = 0, c = \frac{3}{2}, d = 0, k = \frac{13}{24}, f = 0, g = \frac{31}{720}$ and $h = 0$. Thus we obtain the first-order approximate solution

$$u_1(x) = -1 + \frac{3x^2}{2} - \frac{13}{24}x^4 + \frac{31}{720}x^6.$$

Similarly we can find $u_2(x)$ as follows:

$$\begin{aligned} u_2(x) = u_1(x) + \frac{1}{7!}\int_0^x (s-x)^7\{u_1^{(8)}(s) - u_1(s) \\ -8[2s\sin(s) - 7\cos(s)]\}\mathrm{d}s \end{aligned}$$

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Table 4. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one.

x	Exact solution	First-order approximate solution	Absolute error
-1	0	-0.1388888889	$\leq 10^{-2}$
-0.8	-0.2508144153	-0.2505799111	$\leq 10^{-3}$
-0.6	-0.5282147935	-0.5281912000	$\leq 10^{-3}$
-0.4	-0.7736912350	-0.7736903111	$\leq 10^{-5}$
-0.2	-0.9408639147	-0.9408639111	$\leq 10^{-8}$
0	-1	-1	0
0.2	-0.9408639147	-0.9408639111	$\leq 10^{-8}$
0.4	-0.7736912350	-0.7736903111	$\leq 10^{-5}$
0.6	-0.5282147935	-0.5281912000	$\leq 10^{-3}$
0.8	-0.2508144153	-0.2505799111	$\leq 10^{-3}$
1	0	-0.1388888889	$\leq 10^{-2}$

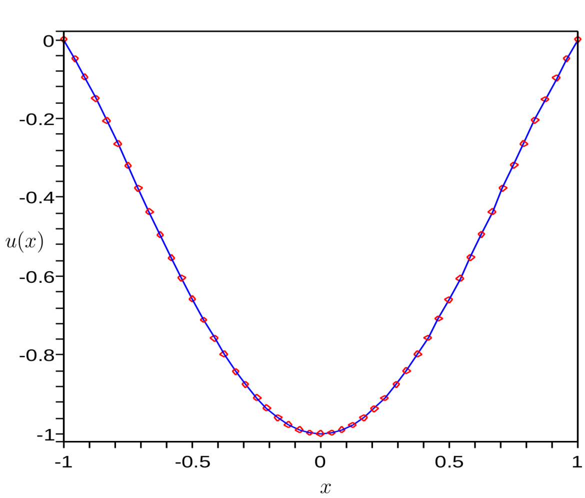

Fig. 4. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one. Solid line, exact solution; symbols, approximate solution.

$$= -1 + \frac{3x^2}{2} - \frac{13x^4}{24} + \frac{31x^6}{720} - \frac{19x^8}{13440} + \frac{13x^{10}}{518400} - \frac{19x^{12}}{68428800} + \frac{61x^{14}}{29059430400} + O(x^{16}).$$

This gives the solution in a closed form by $(x^2-1)\cos(x)$. The first-order approximate solution is of remarkable accuracy. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one is shown in Table 4 and Figure 4.

3.5. Example 5

In order to illustrate the effectiveness of the variational iteration method, we consider the nonlinear boundary-value problem [16]

$$u^{(8)}(x) = \mathrm{e}^{-x}u^2(x), \quad 0 \leq x \leq 1, \tag{12}$$

Table 5. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one.

x	Exact solution	First-order approximate solution	Absolute error
0.1	1.105170918	1.105170918	$\leq 10^{-9}$
0.2	1.221402758	1.221402759	$\leq 10^{-8}$
0.3	1.349858808	1.349858805	$\leq 10^{-8}$
0.4	1.491824698	1.491824681	$\leq 10^{-7}$
0.5	1.648721271	1.648721168	$\leq 10^{-6}$
0.6	1.822118800	1.822118354	$\leq 10^{-6}$
0.7	2.013752707	2.013751158	$\leq 10^{-5}$
0.8	2.225540928	2.225536366	$\leq 10^{-4}$
0.9	2.459603111	2.459591262	$\leq 10^{-3}$
1	e	2.718253968	$\leq 10^{-4}$

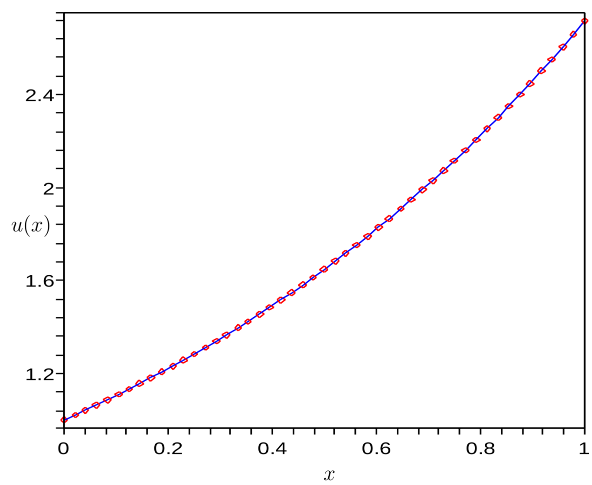

Fig. 5. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one. Solid line, exact solution; symbols, approximate solution.

with the boundary conditions

$$\begin{aligned} &u(0)=1,\ u^{(2)}(0)=1,\ u^{(4)}(0)=1,\ u^{(6)}(0)=1,\\ &u(1)=\mathrm{e},\ u^{(2)}(1)=\mathrm{e},\ u^{(4)}(1)=\mathrm{e},\ u^{(6)}(1)=\mathrm{e}. \end{aligned} \tag{13}$$

According to the variational iteration method we have the following iteration formula:

$$u_{n+1}(x) = u_n(x) + \frac{1}{7!}\int_0^x (s-x)^7[u_n^{(8)}(s) - \mathrm{e}^{-s}u_n^2(s)]\mathrm{d}s. \tag{14}$$

Now, we begin with the arbitrary initial approximation

$$u_0(x) = a + bx + cx^2 + dx^3 + kx^4 + fx^5 + gx^6 + hx^7,$$

where a, b, c, d, k, f, g and h are constants to be determined. By the variational iteration formula (14), we

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obtain $u_1(x)$. Because of exhaustive computations, we do not write $u_1(x)$. Incorporating the boundary conditions (13) into $u_1(x)$ yields a linear system with 8 equations and 8 variables. Solving this linear system simultaneously, we have $a=1$, $b=1$, $c=\frac{1}{2}$, $d=\frac{1}{6}$, $k=\frac{1}{24}$, $f=\frac{1}{120}$, $g=\frac{1}{720}$ and $h=\frac{1}{5040}$. Thus we obtain the first-order approximate solution

$$u_1(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{1}{120}x^5 +\frac{1}{720}x^6+\frac{1}{5040}x^7.$$

As the same, we find $u_2(x)$ as follows:

$$\begin{aligned} u_2(x) &= u_1(x)+\frac{1}{7!}\int_0^x (s-x)^7[u_1^{(8)}(s)-\mathrm{e}^{-s}u_1^2(s)]\mathrm{d}s \\ &= 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{1}{720}x^6 \\ &\quad +\frac{1}{5040}x^7+\frac{1}{40320}x^8+O\left(x^9\right). \end{aligned}$$

This gives the solution in a closed form by e^x. The first-order approximate solution is of remarkable accuracy. Comparison of the first-order approximate solution, $u_1(x)$, with the exact one is shown in Table 5 and Figure 5.

4. Conclusions

We have studied some eighth-order boundary-value problems using the variational iteration method. The initial approximation was selected as a polynomial with unknown constants, which was determined by considering the boundary conditions. The results revealed that the method is remarkably effective. This method is a very promoting one, which will be certainly found widely applied.

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